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Equations of Motion of a Vehicle in a Moving Fluid

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Difficulties with the differing sets of equations used for submersibles, airships, and airplanes are removed by treating the effects of the inertial and added masses as separate functions of the inertial and relative velocities. The equations of motion of a rigid vehicle moving in a perfect fluid are then derived for the case where the fluid mass is accelerating and contains velocity gradients. The classic perfect fluid moments and forces for straight, curved, and convergent flows are recovered. It is shown that the differing sets of equations normally used for submersibles, airships, and aircraft can also be recovered as special cases, but in an augmented form that includes the effects of fluid motion and velocity gradients. In addition, it is shown how the resultant perfect fluid equations may be augmented to include viscous forces and moments derived from other theoretical or experimental sources.

Nomenclature

$\boldsymbol{A}$	= 3 × 3 matrix of center of gravity coordinates, Eq. (17)
$\boldsymbol{A}_e$	= 6 × 6 small perturbation aerodynamic derivative matrix
$\boldsymbol{A}_v$	= vector of nonperfect fluid forces and moments that are a function of the relative velocity $\boldsymbol{x}_r^*$
$\boldsymbol{a}$	= vector from body axis origin to the center of gravity
a_x, a_y, a_z	= body axis coordinates of $\boldsymbol{a}$
$\boldsymbol{B}$	= 3 × 3 matrix of center of buoyancy coordinates, Eq. (17)
$\boldsymbol{b}$	= vector from the body axis origin to the center of buoyancy
b_x, b_y, b_z	= body axis coordinates of $\boldsymbol{b}$
$\boldsymbol{E}$	= direction cosine matrix, Eq. (30)
$\boldsymbol{F}$	= 6 × 1 vector of body axis components of the external forces and moments
$\boldsymbol{F}_d$	= 6 × 1 vector of body axis forces and moments due to rotating axes, the dynamics vector
$\boldsymbol{F}_e$	= steady-state $\boldsymbol{F}$ vector
$\boldsymbol{F}_f$	= 6 × 1 vector of body axis forces and moments due to vessel velocity
$\boldsymbol{I}$	= identity matrix
I_{xx}, I_{yy}, I_{zz}	= roll, pitch, and yaw inertia
I_{xy}, I_{yz}, I_{zx}	= products of inertia
J_{xx}, etc.	= apparent inertia, Eq. (3)
K_0	= kinetic energy of undisturbed circulating fluid mass
L, M, N	= body axis moment components
l_t	= tail moment arm
$\boldsymbol{M}$	= 6 × 6 mass matrix including added masses
M_f	= mass of fluid in multiply connected region
$\boldsymbol{M}_i$	= 6 × 6 vehicle mass matrix, Eq. (9)
$\bar{\boldsymbol{M}}_i$	= 6 × 6 displaced mass matrix, Eq. (14)
$\boldsymbol{M}_r$	= 6 × 6 added mass matrix, Eq. (1)
m	= mass of vehicle
m_x, etc.	= apparent mass, Eq. (3)
$\bar{m}$	= mass of displaced fluid
$\bar{m}_x$, etc.	= apparent displaced mass, Eq. (3)
N_e, E_e, D_e	= north, east, and down vehicle position
$\boldsymbol{P}$	= 6 × 6 matrix of inertial angular velocities, Eq. (39)
p, q, r	= body axis components of inertial angular velocity
p_f, q_f, r_f	= effective rotation rates, Eq. (59)
p_g, q_g, r_g	= body axis components of gust angular velocity
$\boldsymbol{Q}$	= vector of generalized forces
$\boldsymbol{q}$	= vector of problem coordinates, Eq. (24)
$\boldsymbol{R}$	= body angular rates transformation matrix, Eq. (31)
T	= total kinetic energy
$\bar{T}$	= Lagrangian in terms of the generalized coordinates and their velocities
u, v, w	= body axis components of inertial velocity
u_c, v_c, w_c	= body axis components of steady circulating velocity relative to multiply connected region
u_f, v_f, w_f	= body axis components of the inertial velocity of the multiply connected region of fluid
u_g, v_g, w_g	= body axis components of gust velocity
$\dot{v}_c^*$	= apparent rate of change including the flowfield, Eq. (61)
$\boldsymbol{W}$	= 6 × 6 matrix of inertial linear velocities, Eq. (39)
$\boldsymbol{W}_c$	= 6 × 6 matrix of circulating velocities, Eq. (40)
$\boldsymbol{W}_f$	= 6 × 6 matrix of fluid velocities, Eq. (40)
$\boldsymbol{W}_r$	= 6 × 6 matrix of relative velocities, $\boldsymbol{W} - \boldsymbol{W}_f - \boldsymbol{W}_c$
$\dot{w}_c^*$	= apparent rate of change including the flowfield, Eq. (61)
X, Y, Z	= body axis force components
$\boldsymbol{x}$	= inertial linear and angular velocity vector, $(u, v, w, p, q, r)^T$
$\boldsymbol{x}_c$	= circulating velocity vector, $(u_c, v_c, w_c, 0, 0, 0)^T$
$\boldsymbol{x}_f$	= vessel or region velocity vector, $(u_f, v_f, w_f, 0, 0, 0)^T$
$\boldsymbol{x}_g$	= gust or combined fluid and circulating velocity vector, $(u_g, v_g, w_g, p_g, q_g, r_g)^T$
$\boldsymbol{x}_r$	= relative velocity vector of the vehicle and fluid, Eq. (74), $(u_r, v_r, w_r, p_r, q_r, r_r)^T$
$\boldsymbol{x}_r^*$	= relative velocities including the antisymmetric terms from the gradient matrix $\boldsymbol{\Phi}$, Eq. (68)
$[\alpha]^T$	= transformation matrix, Eq. (28)
$[\beta]$	= transformation matrix, Eq. (29)
$\boldsymbol{\delta}$	= small perturbation
$\boldsymbol{\Pi}$	= vector of generalized forces, Eq. (33)
$\boldsymbol{\Phi}$	= 3 × 3 velocity gradient matrix, Eq. (21)
$\boldsymbol{\omega}$	= rate of change of quasi coordinates, Eq. (25)

Subscripts

g	= combined fluid and circulating velocities, Eq. (74)

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$X_{\dot{u}}$ = perfect fluid added masses and inertia, $\partial X/\partial \dot{u}$, etc.
U_{r0}, etc. = steady-state body axis relative velocity

Superscripts

b = referenced to the center of buoyancy
g = referenced to the center of gravity

I. Introduction

THE calculation of the motion of a rigid vehicle in an unsteady heavy fluid is an important part of the analysis and design of many vehicle systems, such as aircraft and submersibles. One of the many problems in computing such motions is that, in a heavy fluid, the motion of the vehicle and that of the fluid are coupled by their respective inertias. To calculate the acceleration of the vehicle we must first establish the fluid pressures around it, but these are in turn dependent not only on the relative velocity between the fluid and the vehicle, but also on the fluid's inertial acceleration and, most important, the acceleration of the vehicle, the very thing we wish to determine. In a perfect fluid no forces are required to maintain steady translation of a vehicle, but if the vehicle is accelerated, then additional forces are needed over and above those needed to accelerate the vehicle in vacuo. These arise as a result of the work done to increase the kinetic energy contained in the fluid when the vehicle is translating at the new speed. In fact, these effects appear as an apparent increase in the mass of the vehicle and are often referred to as added mass. They do not represent fluid mass carried along with the vehicle, but instead represent the additional energy that has to be transferred to the fluid during acceleration.

The basis for analyzing the motion of a rigid vehicle in a perfect fluid was established in the 19th century and is described by Lamb.[1] In this approach, the troublesome calculation of the effects of the fluid pressures on the surface of the solid is avoided by treating the solid and fluid as one dynamical system. Lamb initially considers the case of a single solid moving through an infinite mass of liquid and where the motion of the fluid is entirely due to that of the solid. As such, the motion is irrotational and acyclic. When Lamb considers the impulsive wrench (i.e., the combined force and moment) required to be applied to the solid to generate the motion instantaneously from rest, he shows that it varies in exactly the same way as the momentum of a finite dynamical system, even though the total momentum of the vehicle and fluid is indeterminate.

As part of the analysis Lamb shows that the kinetic energy of the fluid can be expressed as a quadratic form involving the three translational and three rotational velocities of the vehicle. By the use of the modern derivative type notation of naval architecture, the total kinetic energy would be written as

$$T = \frac{1}{2}[u \quad v \quad w \quad p \quad q \quad r]\begin{bmatrix} -X_{\dot{u}} & -X_{\dot{v}} & -X_{\dot{w}} & -X_{\dot{p}} & -X_{\dot{q}} & -X_{\dot{r}} \\ -Y_{\dot{u}} & -Y_{\dot{v}} & -Y_{\dot{w}} & -Y_{\dot{p}} & -Y_{\dot{q}} & -Y_{\dot{r}} \\ -Z_{\dot{u}} & -Z_{\dot{v}} & -Z_{\dot{w}} & -Z_{\dot{p}} & -Z_{\dot{q}} & -Z_{\dot{r}} \\ -L_{\dot{u}} & -L_{\dot{v}} & -L_{\dot{w}} & -L_{\dot{p}} & -L_{\dot{q}} & -L_{\dot{r}} \\ -M_{\dot{u}} & -M_{\dot{v}} & -M_{\dot{w}} & -M_{\dot{p}} & -M_{\dot{q}} & -M_{\dot{r}} \\ -N_{\dot{u}} & -N_{\dot{v}} & -N_{\dot{w}} & -N_{\dot{p}} & -N_{\dot{q}} & -N_{\dot{r}} \end{bmatrix}\begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix} = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{M}_r\boldsymbol{x} \tag{1}$$

The 6×6 matrix is symmetric and, hence, contains 21 unique coefficients. It is usually referred to as the added mass matrix. Lamb uses this, along with Kirchhoff's version of Lagrange's equations of motion, to give the equations of motion of a rigid vehicle in an infinite stationary irrotational medium.

The derivations given by Lamb (the first edition was published in 1879) were used as the basis of the equations of motion of airships, for example, those of Jones and Williams,[2] and culminated in the Williams and Collar[3] report on the loss of the R101. This latter report was almost certainly the first practical, fully nonlinear solution of a multistate flight dynamics problem. A form of the equations of motion suitable for the dynamics of underwater vehicles is given by Lewis et al.[4] These equations were used successfully by Lewis[5] to simulate the motion, in a steady sea, of the SEAPUP remotely operated underwater vehicle. They have also been used to model the motion of airships in a steady uniform atmosphere by Cook et al.[6] and were successfully applied to simulate the motion of the YEZ-2A airship by Nippress and Gomes.[7] Recently, however, the author has had some difficulty in applying these equations to the motion of other vehicles in steady or turbulent winds. In principle, the equations should be applicable to not only underwater vehicles and airships, but also to parafoils and airplanes. Two major problems of the equations in Lewis et al.[4] are that they do not reduce to the small perturbation equations that are used for aircraft in gusts (see Refs. 8 and 9).

II. Difficulties

The equations given by Lewis et al.[4] are of the form

$$\boldsymbol{M}\dot{\boldsymbol{x}} = \boldsymbol{F} + \boldsymbol{F}_d + \boldsymbol{F}_f + \boldsymbol{A} \tag{2}$$

where $\boldsymbol{A}$ is the vector of the fluid dynamics forces and moments due to relative velocity.

From Ref. 4, we define effective or apparent masses and inertias as (the bar refers to the displaced fluid)

$$\begin{aligned} m_x &= m - X_{\dot{u}} & \bar{m}_x &= \bar{m} - X_{\dot{u}} \\ m_y &= m - Y_{\dot{v}} & \bar{m}_y &= \bar{m} - Y_{\dot{v}} \\ m_z &= m - Z_{\dot{w}} & \bar{m}_z &= \bar{m} - Z_{\dot{w}} \\ J_x &= I_x - L_{\dot{p}} & J_{yz} &= I_{yz} + M_{\dot{r}} = I_{zy} + N_{\dot{q}} \\ J_y &= I_y - M_{\dot{q}} & J_{zx} &= I_{zx} + N_{\dot{p}} = I_{xz} + L_{\dot{r}} \\ J_z &= I_z - N_{\dot{r}} & J_{xy} &= I_{xy} + L_{\dot{q}} = I_{yx} + M_{\dot{p}} \end{aligned} \tag{3}$$

If the vehicle center of mass has body axis coordinates of (a_x, a_y, a_z), then the mass matrix is

$$\boldsymbol{M} = \begin{bmatrix} m_x & 0 & 0 & -X_{\dot{p}} & a_z m - X_{\dot{q}} & -a_y m - X_{\dot{r}} \\ 0 & m_y & 0 & -a_z m - Y_{\dot{p}} & -Y_{\dot{q}} & a_x m - Y_{\dot{r}} \\ 0 & 0 & m_z & a_y m - Z_{\dot{p}} & a_x m - Z_{\dot{q}} & -Z_{\dot{r}} \\ -L_{\dot{u}} & -a_z m - L_{\dot{v}} & a_y m - L_{\dot{w}} & J_{xx} & -J_{xy} & -J_{zx} \\ a_z m - M_{\dot{u}} & -M_{\dot{v}} & -a_x m - M_{\dot{w}} & -J_{xy} & J_{yy} & -J_{yz} \\ -a_y m - N_{\dot{u}} & a_x m - N_{\dot{v}} & -N_{\dot{w}} & -J_{zx} & -J_{yz} & J_{zz} \end{bmatrix} \tag{4}$$

the dynamics vector is given by

$$\boldsymbol{F}_d = \begin{bmatrix} -m_z wq + m_y rv + m\left[a_x(q^2 + r^2) - a_y pq - a_z rp\right] \\ -m_x ur + m_z pw + m\left[-a_x pq + a_y(p^2 + r^2) - a_z rq\right] \\ -m_y vp + m_x qu + m\left[-a_x rp - a_y rq + a_z(q^2 + p^2)\right] \\ -(J_z - J_y)rq + J_{yz}(q^2 - r^2) + J_{zx}pq - J_{xy}pr + m[-a_y(vp - qu) + a_z(ur - pw)] \\ -(J_x - J_z)pr - J_{yz}pq + J_{zx}(r^2 - p^2) + J_{xy}qr + m[a_x(vp - qu) - a_z(wq - rv)] \\ -(J_y - J_x)qp + J_{yz}pr - J_{zx}qr + J_{xy}(p^2 - q^2) + m[-a_x(ur - pw) + a_y(wq - rv)] \end{bmatrix} \tag{5}$$

and if the coordinates of the center of buoyancy are (b_x, b_y, b_z), then the fluid motion vector is

$$\boldsymbol{F}_f = \begin{bmatrix} \bar{m}_x \dot{u}_f + \bar{m}_z w_f q - \bar{m}_y r v_f \\ \bar{m}_y \dot{v}_f + \bar{m}_x u_f r - \bar{m}_z p w_f \\ \bar{m}_z \dot{w}_f + \bar{m}_y v_f p - \bar{m}_x q u_f \\ -L_{\dot{u}}\dot{u}_f - (b_z\bar{m} + L_{\dot{v}})\dot{v}_f + (b_y\bar{m} - L_{\dot{w}})\dot{w}_f + \bar{m}[b_y(v_f p - q u_f) - b_z(u_f r - p w_f)] \\ (b_z\bar{m} + M_{\dot{v}})\dot{u}_f - M_{\dot{v}}\dot{v}_f - (b_x\bar{m} - M_{\dot{w}})\dot{w}_f + \bar{m}[-b_x(v_f p - q u_f) + b_z(w_f q - r v_f)] \\ -(b_y\bar{m} + N_{\dot{u}})\dot{u}_f + (b_x\bar{m} - N_{\dot{v}})\dot{v}_f - M_{\dot{w}}\dot{w}_f + \bar{m}[b_x(u_f r - p w_f) - b_y(w_f q - r v_f)] \end{bmatrix} \tag{6}$$

Several difficulties arise with the preceding equations. The most obvious is that if the fluid is unsteady, then the fluid motion vector $\boldsymbol{F}_f$ is a function of the fluid inertial velocity as well as its inertial acceleration, and this is counterintuitive. This can be clearly seen by giving the body the mass and inertia properties of the fluid that it displaces. In that case the relative acceleration between the body and the fluid should be zero, but the preceding equations do not reduce to this. A less obvious difficulty arises with the mass matrix $\boldsymbol{M}$ if we attempt to apply the equations to a vehicle such as a partially constrained dynamic wind-tunnel model. Then it would be expected that the forces and moments due to the vehicle's inertia would depend on its inertial acceleration whereas the forces and moments due to the fluid acceleration (the added mass and inertia terms) would depend on the relative acceleration of the fluid and the vehicle. Thus, we might rearrange the equations in the form

$$\boldsymbol{M}_i\dot{\boldsymbol{x}} + \boldsymbol{M}_r\dot{\boldsymbol{x}}_r = -\boldsymbol{M}_r\dot{\boldsymbol{x}}_f + \boldsymbol{F}_d + \boldsymbol{F}_f + \boldsymbol{A} + \boldsymbol{F} \tag{7}$$

where $\boldsymbol{M}_r$ is given by Eq. (1),

$$\boldsymbol{x}_r = \boldsymbol{x} - \boldsymbol{x}_f \tag{8}$$

are the relative velocities of vehicle and fluid, and the vehicle mass matrix is

$$\boldsymbol{M}_i = \begin{bmatrix} m & 0 & 0 & 0 & ma_z & -ma_y \\ 0 & m & 0 & -ma_z & 0 & ma_x \\ 0 & 0 & m & ma_y & -ma_x & 0 \\ 0 & -ma_z & ma_y & I_{xx} & -I_{xy} & -I_{xz} \\ ma_z & 0 & -ma_x & -I_{xy} & I_{yy} & -I_{yz} \\ -ma_y & ma_x & 0 & -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \tag{9}$$

The fluid motion and fluid acceleration vectors in Eq. (7) can be combined into a new fluid motion vector. This results in most, but not all, of the acceleration terms canceling, and the fluid motion vector still remains a function of the inertial fluid velocity. The source of this is partly because in the derivation used in Lewis et al.[4] some, but not all, perfect fluid relative velocity terms have been absorbed into the vector $\boldsymbol{A}$. As a result, a full set of equations involving all of the perfect fluid terms is required prior to any rearrangement such as that just given.

The equations of motion used for airplanes were first set out by Bryan.[10] He used Euler's version of Newton's equations of motion and expressed the fluid forces and moments as linear functions of the perturbations of the translation and rotational relative velocities from a steady state. In 1921, Cowley and Glauert[11] noted that a downwash derivative $M_{\dot{w}}$ was required to explain the anomalous values observed for the M_q derivative in airplanes. This derivative was due to the convection past the tailplane of the vortex sheet shed by the wing during unsteady motion. Subsequently, workers who were interested in flutter investigated the unsteady aerodynamics of wings (see the review by Lyon[12]) and were able to show that for most flight dynamics situations the unsteady lift effects of the wing could be approximated by an added mass plus a frequency-dependent term. The frequency-dependent term was constant for small values of the reduced frequency typical of airplane dynamics. For most airplanes, these terms are small and are usually ignored. As a result, it is now standard to augment Bryan's model[10] with $Z_{\dot{w}}$ and $M_{\dot{w}}$ derivatives, whose main component is due to the tailplane flying in the lagged downwash field of the main wing.

A major addition to Bryan's equations[10] was the inclusion of atmospheric winds and gusts. This was first detailed by Wilson[13] and details of more modern extensions are given by Etkin[14] and Mulder and van der Vaart.[9] The basis of these descriptions is the assumption that the forces and moments due to gusts arise solely as a result of the relative motion of the vehicle and the fluid. Treating the airplane as a point, the relative fluid velocity can be represented by the difference between the vehicle inertial velocity and the fluid's inertial velocity,

$$\begin{aligned} \boldsymbol{x}_r &= [u_r \quad v_r \quad w_r \quad p_r \quad q_r \quad r_r]^T = \boldsymbol{x} - \boldsymbol{x}_g \\ &= [u \quad v \quad w \quad p \quad q \quad r]^T - [u_g \quad v_g \quad w_g \quad p_g \quad q_g \quad r_g]^T \end{aligned} \tag{10}$$

The p, q, and r terms were included in the gust velocities by Wilson "so as to take into account any possible rotational motion in the gusts."[13] As a result the gust contribution to the forces and moments is represented by

$$\begin{bmatrix} \delta X \\ \delta Y \\ \delta Z \\ \delta L \\ \delta M \\ \delta N \end{bmatrix} = \begin{bmatrix} X_u & X_v & X_w & X_p & X_q & X_r \\ Y_u & Y_v & Y_w & Y_p & Y_q & Y_r \\ Z_u & Z_v & Z_w & Z_p & Z_q & Z_r \\ L_u & L_v & L_w & L_p & L_q & L_r \\ M_u & M_v & M_w & M_p & M_q & M_r \\ N_u & N_v & N_w & N_p & N_q & N_r \end{bmatrix} \begin{bmatrix} \delta u_r \\ \delta v_r \\ \delta w_r \\ \delta p_r \\ \delta q_r \\ \delta r_r \end{bmatrix}$$

$$= \boldsymbol{A}_e\boldsymbol{\delta x}_r = \boldsymbol{A}_e\boldsymbol{x} - \boldsymbol{A}_e\boldsymbol{x}_g \tag{11}$$

A further complication is to recognize the presence of the acceleration derivatives and include in the gust forces and moments additional terms due to the fluid's inertial acceleration. Such terms might be similar to

$$M_r \dot{x}_g \quad (12)$$

but should other terms be included as well?

Later authors such as Stengel,[8] Etkin[15] and Mulder and van der Vaart[9] have gone further and have included the gust penetration effects by representing that the vehicle is, in general, traversing a velocity field containing gradients. This is taken as being equivalent to the vehicle having an apparent rate of rotation. Such apparent rotation rates are then applied to the vehicle aerodynamic model given earlier. A question that then arises is, should the apparent rotation rates be used in the fluid acceleration terms such as Eq. (12)?

The remaining modern extension has been to replace the linear small perturbation model of the aerodynamic forces and moments with full force models, often based on empirical results from wind tunnels, etc. These typically assume that the forces and moments can be written in terms of the relative linear and angular velocities. With such a formulation the question again arises, how should the apparent rates of rotation be used in such a model? Just adding in terms into the equations may not be correct. Additionally, the apparent rotation rates due to the velocity gradients can be due to both irrotational and rotational motion of the fluid, and the method makes no distinction. Finally, the equations of motion contain terms that are the product of the rotation rates and the velocities, and it is not clear whether or not these effective rotation rates should appear at these points in the equations.

In summary, the equations of motion of lifting and buoyant vehicles have developed along similar but different lines ever since their first introduction. They are clearly closely related, but the equations used for airships and underwater vehicles do not reduce to those normally used for airplanes. Both sets of equations were originally derived in a systematic way, but subsequent enhancements have been performed in a rather piecemeal way. The difficulties this has produced have lead to confusion, such as that in the Pretty and Hookway[16] Note questioning airship equations, or complex difficulties as shown in the correspondence between Etkin and D'Eleuterio[17] and Jones and DeLaurier.[18]

There have also been difficulties when applying the equations to nonconventional situations. Recently, for example, the author has been involved with the dynamics of parafoils. These are lifting vehicles of low density, but they have substantial added mass, and it is not clear how to set up their equations of motion for unsteady conditions.

As a result of the preceding difficulties, this paper aims to provide a common set of equations for both buoyant and lifting vehicles in an unsteady fluid and to suggest a more rational means for including empirical data into the equations.

III. Alternative Derivation

As mentioned earlier, Lamb[1] addressed the problem of a vehicle moving in an infinite fluid and was able to show that the impulsive wrench need to start the motion from rest varied in the same way as the momentum of a finite system. He extended the result to the problem of a perforated vehicle moving in fluid contained in a large vessel, which may also be perforated. In that case the fluid is multiply connected, and cyclic motions through the perforations are allowable, independent of the motion of the vehicle. Such circulating motions can provide velocity gradients in the flow. Taking the arguments further, we can apply impulsive forces to the vessel so as to give it, and its fluid contents, a prescribed velocity and acceleration. As a result, the fluid inside the vessel will have velocity gradients plus a bulk velocity and a hydrostatic pressure gradient due to the vessel's acceleration.

The total energy is obtained by applying an approximation used by both Lamb[1] and Taylor,[19] and the equations of motion of a rigid vehicle in a nonuniform flow are derived via Lagrange's equations. The perfect fluid forces and moments in the resulting equations are identified and separated into inertial and relative velocity effects. Finally, gust penetration effects are related to the velocity gradients to represent the variation over the vehicle of the undisturbed moving fluid velocities. These are combined with the perfect fluid terms and the viscous forces and moments that are functions of relative velocity alone.

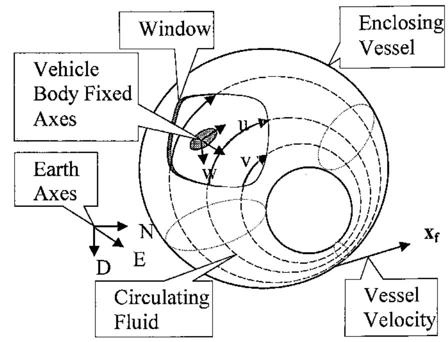

Fig. 1 Vehicle located in a large accelerating vessel containing currents and velocity gradients.

A. Perfect Fluid Equations

Consider a rigid vehicle moving in a perfect fluid that is circulating in a multiply connected vessel, with the vessel itself accelerating. We shall consider a set of axes fixed in the vehicle's body, plus a nonaccelerating set of Earth axes (see Fig. 1). We invoke the approximation used by both Lamb[1] and Taylor[19] that if the vehicle's velocity is made equal to the velocity of the fluid, and its mass is made equal to that of the displaced fluid, then the energy is the same as it would be if the vehicle were absent and the space occupied by fluid. This implies that changes in the circulating fluid velocities over the length of the vehicle are small compared to the velocity of the stream in its neighborhood.

We also define the following quantities evaluated in vehicle body axes:

$$\boldsymbol{x} = (u, v, w, p, q, r)^T$$

which represents the vehicle's velocity relative to the Earth axes,

$$\boldsymbol{x}_f = (u_f, v_f, w_f, 0, 0, 0)^T$$

the vessel's velocity relative to the Earth axes, and

$$\boldsymbol{x}_c = (u_c, v_c, w_c, 0, 0, 0)$$

the steady circulating velocity relative to the vessel that would exist if the vehicle were absent.

We also have, therefore, in Earth axes,

$$\boldsymbol{x}_f = (\dot{N}_f, \dot{E}_f, \dot{D}_f, 0, 0, 0)$$

and we define the relative velocity as

$$\boldsymbol{x}_r = \boldsymbol{x} - \boldsymbol{x}_f - \boldsymbol{x}_c \quad (13)$$

This way of modeling the motion of the fluid is to some extent overdetermined, and we are free to make choices as to how exactly $\boldsymbol{x}_f$ and $\boldsymbol{x}_c$ are chosen. For example, if we are considering motion in a steady current with no gradients, we can set $\boldsymbol{x}_c = 0$. Alternatively, if we are considering no current but with gradients, then we can set $\boldsymbol{x}_f = -\boldsymbol{x}_c$.

We also define the mass matrix of the fluid displaced by the vehicle (strictly speaking, replaced by the vehicle),

$$\bar{\boldsymbol{M}}_i = \begin{bmatrix} \bar{m} & 0 & 0 & 0 & \bar{m}b_z & -\bar{m}b_y \\ 0 & \bar{m} & 0 & -\bar{m}b_z & 0 & \bar{m}b_x \\ 0 & 0 & \bar{m} & \bar{m}b_y & -\bar{m}b_x & 0 \\ 0 & -\bar{m}b_z & \bar{m}b_y & 0 & 0 & 0 \\ \bar{m}b_z & 0 & -\bar{m}b_x & 0 & 0 & 0 \\ -\bar{m}b_y & \bar{m}b_x & 0 & 0 & 0 & 0 \end{bmatrix} \quad (14)$$

The moments and products of inertia in this matrix are zero because the matrix represents the kinetic energy of the fluid that is replaced by the vehicle, and, being irrotational, the fluid has zero rotational energy.

We can then write the Lagrangian of the system, including the bulk translation of the multiply connected fluid, by building up a description of the total kinetic energy via the following steps:

1) Following Lamb (§ 139), we consider the case where the fluid has cyclic irrotational motion through channels in the enclosing

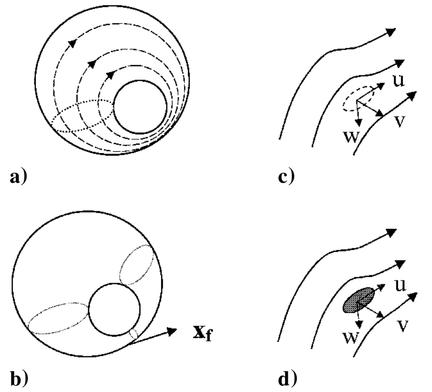

Fig. 2 Stages in developing the Lagrangian.

vessel and assume that it is independent of the motion of the vehicle itself (see Fig. 2a). As shown by Lamb, such cyclic irrotational motion can be impulsively generated from rest via suitable pressures distributed across imaginary diaphragms placed across the channels, so as to render the space singly connected. We shall denote the associated kinetic energy by K_0.

2) Impose an impulsive force on the vessel so as to accelerate it to $\boldsymbol{x}_f$; in this case the additional kinetic energy is $M_f \boldsymbol{x}_f^T \boldsymbol{x}_f / 2$, where M_f is the mass of fluid contained in the vessel (see Fig. 2b).

3) Take the fluid mass that occupies the space that will ultimately be occupied by the vehicle and accelerate it, via a set of impulsive pressures over its surface, to the same speed as the vehicle (see Fig. 2c). In that case, the kinetic energy will increase on two counts, first the increase in kinetic energy of the fluid mass that will be ultimately replaced by the vehicle and second the increase in kinetic energy of the surrounding fluid due to the increased velocity. The latter, of course, can be represented in terms of the added mass. Thus, the extra energy is $\boldsymbol{x}_r^{b^T}(\boldsymbol{M}_r^b + \bar{\boldsymbol{M}}_i^b)\boldsymbol{x}_r^b/2$ where the superscripts b refer to values taken about the center of buoyancy of the object, that is, the center of gravity of the displaced fluid.

4) Finally, we remove the fluid from the space that is to be occupied by the vehicle; this reduces the kinetic energy by $\boldsymbol{x}^{b^T}\bar{\boldsymbol{M}}_i^b\boldsymbol{x}^b/2$ and replaces it with the actual vehicle (see Fig. 2d), thereby increasing the energy by $=\boldsymbol{x}^{g^T}\boldsymbol{M}_i^g\boldsymbol{x}^g/2$, where the superscript g refers to the center of gravity of the vehicle.

In equation form, the total kinetic energy associated with the foregoing process is given by

$$2T = 2K_0 + M_f \boldsymbol{x}_f^T \boldsymbol{x}_f + \boldsymbol{x}_r^{b^T}\left(\boldsymbol{M}_r^b + \bar{\boldsymbol{M}}_i^b\right)\boldsymbol{x}_r^b + \boldsymbol{x}^{g^T}\boldsymbol{M}_i^g\boldsymbol{x}^g - \boldsymbol{x}^{b^T}\bar{\boldsymbol{M}}_i^b\boldsymbol{x}^b \tag{15}$$

The fundamental approximation in Eq. (15) is that $\boldsymbol{x}_r^{b^T}(\boldsymbol{M}_r^b + \bar{\boldsymbol{M}}_i^b)\boldsymbol{x}_r^b/2$ represents the kinetic energy of the fluid replaced by the vehicle. If the circulating fluid has no velocity gradients in the vicinity of the vehicle, then the expression is exact. If velocity gradients are present, then the expression will be approximately correct so long as $(\Delta v/V)^2 \ll 1$, where Δv is the change in circulating fluid velocity over the length of the vehicle and V is the local velocity of the stream. We shall also see later that neither K_0, the energy of the circulating fluid, nor M_f, the mass of circulating fluid, enter into the final form of the equations.

We can relate the velocity vector of the vehicle's center of gravity and center of buoyancy to the velocity vector of the body axis origin by

$$\boldsymbol{x}^g = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{A} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}\boldsymbol{x} \qquad \boldsymbol{x}^b = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}\boldsymbol{x} \tag{16}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & a_z & -a_y \\ -a_z & 0 & a_x \\ a_y & -a_x & 0 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 0 & b_z & -b_y \\ -b_z & 0 & b_x \\ b_y & -b_x & 0 \end{bmatrix} \tag{17}$$

The elements of the added mass matrix change with any change in the position of the axis origin. When we consider the kinetic energy of the fluid as constant, it can be shown that the added mass matrix for an arbitrary axis origin is given by

$$\boldsymbol{M}_r = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{B} & \boldsymbol{I} \end{bmatrix}\boldsymbol{M}_r^b\begin{bmatrix} \boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \tag{18}$$

Failure to bear this in mind can lead to serious errors, as will be shown in Sec. III.B.

Now the relative velocity between the fluid and the vehicle at the vehicle's center of buoyancy is given by

$$\boldsymbol{x}_r^b = \boldsymbol{x}^b - \boldsymbol{x}_f - \boldsymbol{x}_c^b \tag{19}$$

Because of the velocity gradients in the circulating fluid, the circulating velocity that would exist at the center of buoyancy if the vehicle were absent is different from that which would exist at the body axis origin. As a result, we can write

$$\boldsymbol{x}_c^b = \boldsymbol{x}_c + \begin{bmatrix} \boldsymbol{\Phi}^T \\ \boldsymbol{0} \end{bmatrix}\boldsymbol{b} \tag{20}$$

and

$$\boldsymbol{\Phi} = \begin{bmatrix} \dfrac{\partial u_c}{\partial x} & \dfrac{\partial v_c}{\partial x} & \dfrac{\partial w_c}{\partial x} \\ \dfrac{\partial u_c}{\partial y} & \dfrac{\partial v_c}{\partial y} & \dfrac{\partial w_c}{\partial y} \\ \dfrac{\partial u_c}{\partial z} & \dfrac{\partial v_c}{\partial z} & \dfrac{\partial w_c}{\partial z} \end{bmatrix} \tag{21}$$

Because the flow is irrotational the preceding matrix is symmetric. Substituting Eqs. (16) and (20) into Eq. (19) gives

$$\boldsymbol{x}_r^b = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}\left(\boldsymbol{x}_r - \begin{bmatrix} \boldsymbol{\Phi}^T \\ \boldsymbol{0} \end{bmatrix}\boldsymbol{b}\right) \tag{22}$$

The Lagrangian can now be written as

$$2T = 2K_0 + M_f \boldsymbol{x}_f^T \boldsymbol{x}_f + \left(\boldsymbol{x}_r - \begin{bmatrix} \boldsymbol{\Phi}^T \\ \boldsymbol{0} \end{bmatrix}\boldsymbol{b}\right)^T (\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i) \times \left(\boldsymbol{x}_r - \begin{bmatrix} \boldsymbol{\Phi}^T \\ \boldsymbol{0} \end{bmatrix}\boldsymbol{b}\right) + \boldsymbol{x}^T(\boldsymbol{M}_i - \bar{\boldsymbol{M}}_i)\boldsymbol{x} \tag{23}$$

where all of the items are referred to the body axis origin.

The vehicle equations of motion with the extra terms for the fluid's motion can be derived using Lagrange's equations. However, the preceding Lagrangian is not given in terms of generalized coordinates and their corresponding velocities, and so the conventional form of Lagrange's equations cannot be used.

This problem belongs to a class that can be solved by the use of quasi coordinates, and this general class of problems is described by Whittaker[20] and Meirovitch.[21] The true coordinates of the current problem can be taken as

$$\boldsymbol{q} = (N_e, E_e, D_e, \phi, \theta, \psi, N_f, E_f, D_f, N_c, E_c, D_c) \tag{24}$$

whereas the rates of change of the corresponding quasi coordinates are

$$\boldsymbol{\omega} = (u, v, w, p, q, r, u_f, v_f, w_f, u_c, v_c, w_c) \tag{25}$$

Lagrange's equations in matrix form are

$$\dot{\bar{\boldsymbol{T}}}_{\dot{q}} - \bar{\boldsymbol{T}}_q = \boldsymbol{Q} \tag{26}$$

where

$$\bar{\boldsymbol{T}}_q = \begin{bmatrix} \dfrac{\partial \bar{T}}{\partial N_e} & \dfrac{\partial \bar{T}}{\partial E_e} & \dfrac{\partial \bar{T}}{\partial D_e} & \dfrac{\partial \bar{T}}{\partial \phi} & \dfrac{\partial \bar{T}}{\partial \theta} & \dfrac{\partial \bar{T}}{\partial \psi} & \dfrac{\partial \bar{T}}{\partial N_f} & \dfrac{\partial \bar{T}}{\partial E_f} & \dfrac{\partial \bar{T}}{\partial D_f} & \dfrac{\partial \bar{T}}{\partial N_c} & \dfrac{\partial \bar{T}}{\partial E_c} & \dfrac{\partial \bar{T}}{\partial D_c} \end{bmatrix}^T \tag{27}$$

and the Lagrangian $\bar{T} = \bar{T}(\dot{q}\ q)$ is expressed in terms of the generalized coordinates and their velocities, and $\boldsymbol{Q}$ represents the generalized forces. To convert this to quasi coordinate first express the rates of change of the quasi coordinates as linear combinations of the rates of change of the true coordinates,

$$\boldsymbol{\omega} = [\alpha]^T \dot{\boldsymbol{q}} \tag{28}$$

Then the inverse relation is

$$\dot{\boldsymbol{q}} = [\beta]\boldsymbol{\omega} \tag{29}$$

For the present problem we have

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{N}_e \\ \dot{E}_e \\ \dot{D}_e \end{bmatrix} \tag{30}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & \sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{31}$$

plus equations similar to Eq. (30) for $[u_c\ v_c\ w_c]^T$ and $[u_f\ v_f\ w_f]^T$.

The preceding equations define the $[\alpha]$ matrix:

$$[\alpha]^T = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{E} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{E} \end{bmatrix} \tag{32}$$

and the 3×3 matrices $\boldsymbol{E}$ and $\boldsymbol{R}$ are those of Eqs. (30) and (31), respectively.

Premultiply Lagrange's equation by $[\beta]^T$ to give

$$[\beta]^T[\dot{\bar{\boldsymbol{T}}}_{\dot{q}} - \bar{\boldsymbol{T}}_q] = [\beta]^T \boldsymbol{Q} = \Pi \tag{33}$$

and we can write

$$\bar{\boldsymbol{T}}_{\dot{q}} = \boldsymbol{\omega}\left[\frac{\partial}{\partial \dot{\boldsymbol{q}}}\right]^T \boldsymbol{T}_\omega = [\alpha]^T \boldsymbol{T}_\omega \tag{34}$$

where $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{\omega}, \boldsymbol{q})$ is the Lagrangian expressed solely in terms of the generalized co-ordinates and the quasi-coordinate velocities. Therefore,

$$[\beta]^T[[\alpha]\dot{\boldsymbol{T}}_\omega + [\dot{\alpha}]\boldsymbol{T}_\omega - \dot{\boldsymbol{T}}_q] = \Pi \tag{35}$$

but

$$\bar{\boldsymbol{T}}_q = \boldsymbol{T}_q + \boldsymbol{\omega}\left[\frac{\partial}{\partial \boldsymbol{q}}\right]^T \boldsymbol{T}_\omega \tag{36}$$

and writing

$$[\beta]^T \boldsymbol{T}_q = \boldsymbol{T}_\pi \tag{37}$$

gives

$$\dot{\boldsymbol{T}}_\omega + [\beta]^T\left([\dot{\alpha}] - \omega\left[\frac{\partial}{\partial \boldsymbol{q}}\right]\right)\boldsymbol{T}_\omega - \boldsymbol{T}_\pi = \Pi \tag{38}$$

After considerable algebraic manipulation and the choice of following matrices

$$\boldsymbol{P} = \begin{bmatrix} 0 & -r & q & 0 & 0 & 0 \\ r & 0 & -p & 0 & 0 & 0 \\ -q & p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -r & q \\ 0 & 0 & 0 & r & 0 & -p \\ 0 & 0 & 0 & -q & p & 0 \end{bmatrix} \qquad \boldsymbol{W} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -w & v & 0 & 0 & 0 \\ w & 0 & -u & 0 & 0 & 0 \\ -v & u & 0 & 0 & 0 & 0 \end{bmatrix} \tag{39}$$

$\boldsymbol{W}_r$, $\boldsymbol{W}_f$, and $\boldsymbol{W}_c$, are similar, but using $\boldsymbol{x}_r$, $\boldsymbol{x}_f$, and $\boldsymbol{x}_c$ the foregoing expressions yield the Lagrangian equations of motion,

$$\dot{\boldsymbol{T}}_x + (\boldsymbol{P} + \boldsymbol{W})\boldsymbol{T}_x + \boldsymbol{W}_f \boldsymbol{T}_f + \boldsymbol{W}_c \boldsymbol{T}_c - \boldsymbol{T}_s = \boldsymbol{F} \tag{40}$$

where,

$$\boldsymbol{T}_f = \begin{bmatrix} \dfrac{\partial T}{\partial u_f} & \dfrac{\partial T}{\partial v_f} & \dfrac{\partial T}{\partial w_f} & 0 & 0 & 0 \end{bmatrix}^T$$

$$\boldsymbol{T}_c = \begin{bmatrix} \dfrac{\partial T}{\partial u_c} & \dfrac{\partial T}{\partial v_c} & \dfrac{\partial T}{\partial w_c} & 0 & 0 & 0 \end{bmatrix}^T \tag{41}$$

$$\boldsymbol{T}_x = \begin{bmatrix} \dfrac{\partial T}{\partial u} & \dfrac{\partial T}{\partial v} & \dfrac{\partial T}{\partial w} & \dfrac{\partial T}{\partial p} & \dfrac{\partial T}{\partial q} & \dfrac{\partial T}{\partial r} \end{bmatrix}^T$$

$$\boldsymbol{T}_s = \begin{bmatrix} \dfrac{\partial T}{\partial x} & \dfrac{\partial T}{\partial y} & \dfrac{\partial T}{\partial z} & \dfrac{\partial T}{\partial \xi} & \dfrac{\partial T}{\partial \eta} & \dfrac{\partial T}{\partial \zeta} \end{bmatrix}^T \tag{42}$$

and x, y, z, ξ, η, and ζ are linear displacements and angular deflections about the body axes.

The equations of motion (40) can be written as

$$\dot{\boldsymbol{T}}_x = \boldsymbol{F} - (\boldsymbol{P} + \boldsymbol{W})\boldsymbol{T}_x - \boldsymbol{W}_f \boldsymbol{T}_f - \boldsymbol{W}_c \boldsymbol{T}_c - \boldsymbol{T}_s \tag{43}$$

These equations are similar to Lamb's, but with fluid motion and velocity gradient terms included. They consist of six equations, although the problem has 12 degrees of freedom. The remaining six equations are the equations of motion of the multiply connected region. For the present problem, the motion of the region is regarded as prescribed and, hence, the equations are not required.

Differentiating the Lagrangian (20) and remembering that $\boldsymbol{x}_r = \boldsymbol{x} - \boldsymbol{x}_f - \boldsymbol{x}_c$ gives

$$\boldsymbol{T}_x = (\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\left(\boldsymbol{x} - \boldsymbol{x}_f - \boldsymbol{x}_c - \begin{bmatrix} \boldsymbol{\Phi}^T \\ \boldsymbol{0} \end{bmatrix}\boldsymbol{b}\right) + (\boldsymbol{M}_i - \bar{\boldsymbol{M}}_i)\boldsymbol{x}$$

$$\dot{\boldsymbol{T}}_x = (\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)(\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_f) + (\boldsymbol{M}_i - \bar{\boldsymbol{M}}_i)\dot{\boldsymbol{x}} = (\boldsymbol{M}_r + \boldsymbol{M}_i)\dot{\boldsymbol{x}} - (\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\dot{\boldsymbol{x}}_f$$

$$\boldsymbol{T}_f = \boldsymbol{M}_f \boldsymbol{x}_f - \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}(\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\left(\boldsymbol{x} - \boldsymbol{x}_f - \boldsymbol{x}_c - \begin{bmatrix} \boldsymbol{\Phi}^T \\ \boldsymbol{0} \end{bmatrix}\boldsymbol{b}\right)$$

$$\boldsymbol{T}_c = -\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}(\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\left(\boldsymbol{x} - \boldsymbol{x}_f - \boldsymbol{x}_c - \begin{bmatrix} \boldsymbol{\Phi}^T \\ \boldsymbol{0} \end{bmatrix}\boldsymbol{b}\right) \tag{44}$$

Substituting in Eq. (43) gives

$$\begin{aligned} (\boldsymbol{M}_r + \boldsymbol{M}_i)\dot{\boldsymbol{x}} - (\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\dot{\boldsymbol{x}}_f &= \boldsymbol{F} - (\boldsymbol{P} + \boldsymbol{W})(\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\left(\boldsymbol{x}_r - \begin{bmatrix} \boldsymbol{\Phi}^T \\ \boldsymbol{0} \end{bmatrix}\boldsymbol{b}\right) - (\boldsymbol{P} + \boldsymbol{W}) \\ &\quad \times (\boldsymbol{M}_i + \bar{\boldsymbol{M}}_i)\boldsymbol{x} + \boldsymbol{W}_f(\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\left(\boldsymbol{x}_r - \begin{bmatrix} \boldsymbol{\Phi}^T \\ \boldsymbol{0} \end{bmatrix}\boldsymbol{b}\right) \\ &\quad + \boldsymbol{W}_c(\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\left(\boldsymbol{x}_r - \begin{bmatrix} \boldsymbol{\Phi}^T \\ \boldsymbol{0} \end{bmatrix}\boldsymbol{b}\right) + \boldsymbol{T}_s \end{aligned} \tag{45}$$

or writing $\boldsymbol{W}_r = \boldsymbol{W} - \boldsymbol{W}_{\bar{f}} - \boldsymbol{W}_c$ gives the equations of motion as

$$(\boldsymbol{M}_r + \boldsymbol{M}_i)\dot{\boldsymbol{x}} = \boldsymbol{F} - (\boldsymbol{P} + \boldsymbol{W})(\boldsymbol{M}_i - \bar{\boldsymbol{M}}_i)\boldsymbol{x} + (\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\dot{\boldsymbol{x}}_f - (\boldsymbol{P} + \boldsymbol{W}_r)(\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\left(\boldsymbol{x}_r - \begin{bmatrix}\boldsymbol{\Phi}^T \\ \boldsymbol{0}\end{bmatrix}\boldsymbol{b}\right) + \boldsymbol{T}_s \tag{46}$$

The remaining term to be considered is $\boldsymbol{T}_s$: This is the change in energy resulting from quasi-coordinate displacements, that is, displacements along and around the body axes. The corresponding forces and moments arise from the velocity gradients in the circulating fluid. The only component of the Lagrangian that contributes is

$$\left(\boldsymbol{x}_r - \begin{bmatrix}\boldsymbol{\Phi}^T \\ \boldsymbol{0}\end{bmatrix}\boldsymbol{b}\right)^T (\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\left(\boldsymbol{x}_r - \begin{bmatrix}\boldsymbol{\Phi}^T \\ \boldsymbol{0}\end{bmatrix}\boldsymbol{b}\right) \tag{47}$$

and we have from Eq. (22)

$$\left(\boldsymbol{x}_r - \begin{bmatrix}\boldsymbol{\Phi}^T \\ \boldsymbol{0}\end{bmatrix}\boldsymbol{b}\right) = \begin{bmatrix}\boldsymbol{I} & -\boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix}\boldsymbol{x}_r^b \tag{48}$$

As a result, expression (47) can be written as

$$\boldsymbol{x}_r^{b^T}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{I}\end{bmatrix}(\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\begin{bmatrix}\boldsymbol{I} & -\boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix}\boldsymbol{x}_r^b \tag{49}$$

or

$$\left(\boldsymbol{x}^b - \boldsymbol{x}_f - \boldsymbol{x}_c^b\right)^T\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{I}\end{bmatrix}(\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\begin{bmatrix}\boldsymbol{I} & -\boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix}\left(\boldsymbol{x}^b - \boldsymbol{x}_f - \boldsymbol{x}_c^b\right) \tag{50}$$

Differentiating Eq. (50) with respect to the quasi coordinates gives

$$\boldsymbol{T}_s = -\begin{bmatrix}\boldsymbol{\Phi} & \boldsymbol{0} \\ -\boldsymbol{B\Phi} & \boldsymbol{0}\end{bmatrix}(\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\left(\boldsymbol{x}_r - \begin{bmatrix}\boldsymbol{\Phi}^T \\ \boldsymbol{0}\end{bmatrix}\boldsymbol{b}\right) \tag{51}$$

As a result, the final form of the equations of motion becomes

$$(\boldsymbol{M}_r + \boldsymbol{M}_i)\dot{\boldsymbol{x}} = \boldsymbol{F} - (\boldsymbol{P} + \boldsymbol{W})(\boldsymbol{M}_i - \bar{\boldsymbol{M}}_i)\boldsymbol{x} + (\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\dot{\boldsymbol{x}}_f - (\boldsymbol{P} + \boldsymbol{W}_r)(\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\left(\boldsymbol{x}_r - \begin{bmatrix}\boldsymbol{\Phi}^T \\ \boldsymbol{0}\end{bmatrix}\boldsymbol{b}\right) - \begin{bmatrix}\boldsymbol{\Phi} & \boldsymbol{0} \\ -\boldsymbol{B\Phi} & \boldsymbol{0}\end{bmatrix}(\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\left(\boldsymbol{x}_r - \begin{bmatrix}\boldsymbol{\Phi}^T \\ \boldsymbol{0}\end{bmatrix}\boldsymbol{b}\right) \tag{52}$$

It is instructive to examine these equations. The fluid motion terms $(\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\dot{\boldsymbol{x}}_f$ depend only on the fluid's inertial acceleration and do not depend on the fluid's inertial velocity, thus solving one of the problems outlined in Sec. II. In addition, both the added mass of the vehicle and the mass of displaced fluid are involved in the expression. The mass matrix is the conventional $(\boldsymbol{M}_r + \boldsymbol{M}_i)$, and the dynamics vector $-(\boldsymbol{P} + \boldsymbol{W})(\boldsymbol{M}_i - \bar{\boldsymbol{M}}_i)\boldsymbol{x}$ (that is, terms due to rotating body axes) involves the difference between the vehicle's mass and that of the displaced fluid. Unlike the case of the equations in Sec. II, giving the vehicle the same mass and inertias as the displaced fluid results in the relative acceleration becoming zero, as would be expected.

The final two terms in Eq. (52) are the perfect fluid forces and moments that are a function of the relative velocity and the velocity gradients alone. They can be combined and expressed in terms of the relative velocities at the center of buoyancy to give

$$-\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{B} & \boldsymbol{I}\end{bmatrix}\left(\boldsymbol{P} + \boldsymbol{W}_r^b + \begin{bmatrix}\boldsymbol{\Phi} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\right)\left(\boldsymbol{M}_r^b + \bar{m}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\right)\boldsymbol{x}_r^b \tag{53}$$

where the superscript b implies quantities evaluated at the center of buoyancy.

Equation (53) represents a set of forces and moments that act at the center of buoyancy and that are transformed to the origin by the left-hand matrix. If the matrices in the preceding expression are partitioned into 3 × 3 sub matrices (written in nonbold type) then the equations of motion become

$$(\boldsymbol{M}_r + \boldsymbol{M}_i)\dot{\boldsymbol{x}} = \boldsymbol{F} - (\boldsymbol{P} + \boldsymbol{W})(\boldsymbol{M}_i - \bar{\boldsymbol{M}}_i)\boldsymbol{x} + (\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\dot{\boldsymbol{x}}_f - \begin{bmatrix}\boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{B} & \boldsymbol{I}\end{bmatrix}\begin{bmatrix}(P + \Phi)\left(\bar{m}I + M_{11}^b\right) & (P + \Phi)M_{12}^b \\ W_r^b M_{11}^b + PM_{12}^b & W_r^b M_{12}^b + PM_{22}^b\end{bmatrix}\boldsymbol{x}_r^b \tag{54}$$

If the mass of displaced fluid is negligible and the gradients are zero, Eq. (52) reverts to the conventional airplane equations of motion:

$$\boldsymbol{M}\dot{\boldsymbol{x}} = \boldsymbol{F} - (\boldsymbol{P} + \boldsymbol{W})\boldsymbol{M}\boldsymbol{x} \tag{55}$$

B. Gust Penetration Effects

The terms involving Φ represent the perfect fluid effects of velocity gradients. With the axes at the center of buoyancy, the moments due to the gradients are zero and the forces are

$$\boldsymbol{\Phi}\begin{bmatrix} u_r(\bar{m} - X_{\dot{u}}) - v_r X_{\dot{v}} - w_r X_{\dot{w}} - pL_{\dot{u}} - qM_{\dot{u}} - rN_{\dot{u}} \\ -u_r Y_{\dot{u}} + v_r(\bar{m} - Y_{\dot{v}}) - w_r Y_{\dot{w}} - pL_{\dot{v}} - qM_{\dot{v}} - rN_{\dot{v}} \\ -u_r Z_{\dot{u}} - v_r Z_{\dot{v}} + w_r(\bar{m} - Z_{\dot{w}}) - pL_{\dot{w}} - qM_{\dot{w}} - rN_{\dot{w}} \end{bmatrix} \tag{56}$$

If we consider a stationary axisymmetric vehicle with no rotation, the x force component becomes

$$X = u_r\left\{(\bar{m} - X_{\dot{u}})\frac{\partial u}{\partial x} - X_{\dot{v}}\frac{\partial v}{\partial x} - X_{\dot{w}}\frac{\partial w}{\partial x}\right\} \tag{57}$$

which is the same result as that obtained by Lamb[1] and Taylor.[19]

If the vehicle is moving in a steady but nonuniform stream, there will be time-varying relative velocity components due to the vehicle's translation through the fluid gradients. The perfect fluid effects of these gradients are already included in the equations as shown earlier, but in many situations such as wind shear and turbulence the velocity field will not be irrotational. In this case the matrix of gradients Φ can be split[22] into symmetric and antisymmetric parts, with the symmetric representing the irrotational strain rates [used in Eqs. (52) or (54)] and the antisymmetric representing vorticity. The effects of such vortical gradients is not included in the preceding equations. A clue as to how to handle the vortical components is given by observing that the upper-left submatrix of $\boldsymbol{P}$ is antisymmetric in p, q, and r and that it combines linearly with $\boldsymbol{\Phi}$ in Eq. (54). This suggests that any skew symmetric part of $\boldsymbol{\Phi}$ can be treated as similar to $\boldsymbol{P}$. In effect, such vortical velocity gradients can be treated as effective rotation rates because they produce velocity distributions similar to those due to rotation,

$$[p_f \quad q_f \quad r_f]^T = \begin{bmatrix}\dfrac{\partial w_c}{\partial y} - \dfrac{\partial v_c}{\partial z} & \dfrac{\partial u_c}{\partial z} - \dfrac{\partial w_c}{\partial x} & \dfrac{\partial v_c}{\partial x} - \dfrac{\partial u_c}{\partial y}\end{bmatrix}^T = (\Phi - \Phi^T) \tag{58}$$

These effective rotation rates can be carefully applied to the vehicle's rotary derivatives. Care must be taken, however, because not all of the gradients have the same influence. In an airplane, for example, the vertical dimensions are small compared to the tail arm and the span, and so the rotary derivatives are due mainly to lift associated with changes in the incidence of the tailplane and the wing tips, that is, the result of x and y gradients. As a result, the variation with z should be ignored, so that

$$[p_f \quad q_f \quad r_f]^T = \left[\frac{\partial w_c}{\partial y}, \quad -\frac{\partial w_c}{\partial x}, \quad \frac{\partial v_c}{\partial x} - \frac{\partial u_c}{\partial y}\right]^T \tag{59}$$

This is the linear field approximation used by Etkin.[15]

If the vehicle has a velocity U (along the x axis) relative to the fluid,

$$[p_f \quad q_f \quad r_f]^T = \left[\frac{\partial w_c}{\partial y}, \quad -\frac{\dot{w}_c^*}{U}, \quad \frac{\dot{v}_c^*}{U} - \frac{\partial u_c}{\partial y}\right]^T \tag{60}$$

where the starred terms are the apparent rate of change due to the motion of the turbulence field past the vehicle, not the inertial accelerations of the undisturbed fluid. The preceding expression is added to the vehicle rotation rates to give the effective relative angular velocity for use in the aerodynamic calculations (it being solely a function of the relative velocities). This avoids one of the problems mentioned earlier with the formulation used in Ref. 4, which had fluid velocity terms in the fluid motion vector and, which made it unclear as to how such effective rotation rates could be included in the equations. As mentioned earlier, Eqs. (52) and (54) have no such terms in the fluid motion vector.

nature of the $\dot{\alpha}$ derivatives are discussed by Etkin[14] and Hancock,[23] and whereas the preceding assumption is not strictly correct, it is still a necessary working approximation for many situations.

C. Use of External Fluid Dynamic Data

The last term in Eq. (54) is a function of the relative velocities only and could, therefore, be absorbed into the vector $\boldsymbol{F}$ (external forces and moments) as the perfect fluid components due to the relative velocity between the fluid and the vehicle. In many cases the elements of the vector will be derived empirically from wind-tunnel or tank facilities and will include some or all of the perfect fluid effects. In addition, parts of $\boldsymbol{F}$ may be determined by analytical and computational models of the relative velocity effects. These in turn may contain perfect fluid components. The nongradient perfect fluid relative velocity terms that could be absorbed into the fluid dynamic vector are (the b superscript has been dropped in the large matrix)

$$-\left(\boldsymbol{P} + \boldsymbol{W}_r^b\right)\left(\boldsymbol{M}_r^b + \bar{m}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\right)\boldsymbol{x}_r^b$$

$$= \left\{\begin{matrix}
-rY_{\dot u} + qZ_{\dot u} & -r(Y_{\dot v} - \bar m) + qZ_{\dot v} & -rY_{\dot w} + q(Z_{\dot w} - \bar m) & -rY_{\dot p} + qZ_{\dot p} & -rY_{\dot q} + qZ_{\dot q} & -rY_{\dot r} + qZ_{\dot r} \\
r(X_{\dot u} - \bar m) - pZ_{\dot u} & rX_{\dot v} - pZ_{\dot v} & rX_{\dot w} - p(Z_{\dot w} - \bar m) & rX_{\dot p} - pZ_{\dot p} & rX_{\dot q} - pZ_{\dot q} & rX_{\dot r} - pZ_{\dot r} \\
-q(X_{\dot u} - \bar m) + pY_{\dot u} & -qX_{\dot v} + p(Y_{\dot v} - \bar m) & -qX_{\dot w} + pY_{\dot w} & -qX_{\dot p} + pY_{\dot p} & -qX_{\dot q} + pY_{\dot q} & -qX_{\dot r} + pY_{\dot r} \\
-w_rY_{\dot u} + v_rZ_{\dot u} - rM_{\dot u} + qN_{\dot u} & -w_rY_{\dot v} + v_rZ_{\dot v} - rM_{\dot v} + qN_{\dot v} & -w_rY_{\dot w} + v_rZ_{\dot w} - rM_{\dot w} + qN_{\dot w} & -w_rY_{\dot p} + v_rZ_{\dot p} - rM_{\dot p} + qN_{\dot p} & -w_rY_{\dot q} + v_rZ_{\dot q} - rM_{\dot q} + qN_{\dot q} & -w_rY_{\dot r} + v_rZ_{\dot r} - rM_{\dot r} + qN_{\dot r} \\
w_rX_{\dot u} - u_rZ_{\dot u} + rL_{\dot u} - pN_{\dot u} & w_rX_{\dot v} - u_rZ_{\dot v} + rL_{\dot v} - pN_{\dot v} & w_rX_{\dot w} - u_rZ_{\dot w} + rL_{\dot w} - pN_{\dot w} & w_rX_{\dot p} - u_rZ_{\dot p} + rL_{\dot p} - pN_{\dot p} & w_rX_{\dot q} - u_rZ_{\dot q} + rL_{\dot q} - pN_{\dot q} & w_rX_{\dot r} - u_rZ_{\dot r} + rL_{\dot r} - pN_{\dot r} \\
-v_rX_{\dot u} + u_rY_{\dot u} - qL_{\dot u} + pM_{\dot u} & -v_rX_{\dot v} + u_rY_{\dot v} - qL_{\dot v} + pM_{\dot v} & -v_rX_{\dot w} + u_rY_{\dot w} - qL_{\dot w} + pM_{\dot w} & -v_rX_{\dot p} + u_rY_{\dot p} - qL_{\dot p} + pM_{\dot p} & -v_rX_{\dot q} + u_rY_{\dot q} - qL_{\dot q} + pM_{\dot q} & -v_rX_{\dot r} + u_rY_{\dot r} - qL_{\dot r} + pM_{\dot r}
\end{matrix}\right\}\boldsymbol{x}_r^b \tag{62}$$

The distinction between the real and apparent fluid accelerations is important because, in some derivations,[9] the use of the frozen turbulence approximation leads to them being lumped together so that

$$Z_{\dot w_f} = Z_{\dot w} - Z_q / l_t \tag{61}$$

In the case of a moving sea or nonfrozen turbulence, this is incorrect and the preceding treatment keeps them distinct. Thus, the problems outlined in Sec. II can be avoided.

One important point is that the linear field approximation must break down at some point as the wavelength of the disturbance reduces. This applies to both the perfect fluid effects and the vortical effects. The basic approximation used to establish the kinetic energy was that if the vehicle velocity was made equal to the velocity of the fluid, and its mass was made equal to that of the displaced fluid, then the energy was the same as that which would have existed had the vehicle been absent and the space occupied by fluid. This implies that $(\Delta v / V)^2 \ll 1$, where Δv is the change in the circulating fluid velocities over the length of the vehicle and V is the velocity of the stream. If this is not so, then more extensive models may be needed at this point depending on the application.

The acceleration-dependent terms or added masses arise from the work done in accelerating the perfect fluid. In a real fluid, additional acceleration effects such as the increase in vorticity and its convection past tailplanes, etc., come into play. It is assumed that all such effects are added into the corresponding perfect fluid added mass terms. This is a reasonable assumption for streamlined vehicles supported by buoyancy, but is less exact for vehicles with substantial lift such as airplanes. The unsteady aspects of lift generation and the

Dependent on how the elements of the $\boldsymbol{F}$ vector are determined, some or all of the preceding terms may already be included, and care must be exercised that terms are neither omitted nor accounted for twice. The risk of double accounting can be seen by considering an axisymmetric vehicle in a turn with $w = p = q = 0$. Then the perfect fluid yawing moment is, from Eq. (62),

$$N = -v_r X_{\dot u} u_r + u_r Y_{\dot v} v_r \tag{63}$$

Now, if the vehicle is sideslipping at an angle β,

$$\sin\beta = v_r / V_{\text{tot}} \qquad \cos\beta = u_r / V_{\text{tot}}$$

Hence,

$$N = \left(V_{\text{tot}}^2 \big/ 2\right)(Y_{\dot v} - X_{\dot u}) \sin 2\beta \tag{64}$$

that is, this is the classic perfect fluid moment given by Munk,[24] and depending on the source of the data this may already be included in the vector $\boldsymbol{F}$. This danger was noted by Lingard,[25] who derived a set of equations for the longitudinal motion of a parafoil and rightly noted that previous investigators had erroneously left terms like Eq. (63) in the equations of motion, while at the same time using wind-tunnel data for the pitching moment, thereby double accounting. Lingard removed the terms from his equations, but without correcting the rest of the added mass matrix for axes not at the center of buoyancy (as described earlier in Sec. III.A). As a result, Lingard's equations predict a motion that is dependent on the chosen axis position, which is clearly a physically incorrect result. Replacing Lingard's equations with those just derived removes this problem.

In the same side-slipping case there is also a side force from Eq. (62),

$$Y = u_r r(X_{\dot{u}} - \bar{m}) - v_r r X_{\dot{v}} \tag{65}$$

and any empirical data used to calculate the side force, such as whirling arm results, may or may not have been adjusted for this centrifugal term.

As can be seen from the preceding examples, the absorption of the perfect fluid terms into $\boldsymbol{F}$ has to be performed with considerable care, and the details will vary depending on the application and the nature of the data used for the $\boldsymbol{F}$ vector. The choice is not arbitrary and must be performed correctly.

IV. Small Perturbation Equations

Assume that the steady wind components are zero and that there is no initial sideslip. The vector $\boldsymbol{F}$ in Eq. (52) is the vector of external forces and moments and, in this case, we will expand it to

$$\boldsymbol{F} = \boldsymbol{F}_e + \boldsymbol{A}_v\left\{\boldsymbol{x}_r^*\right\} \tag{66}$$

where $\boldsymbol{A}_v\{\,\}$ is a vector function giving the nonperfect fluid forces and moments that are a function of the relative velocity $\boldsymbol{x}_r^*$, where

$$\boldsymbol{x}_r^* = (u_r, v_r, w_r, p - p_f, q - q_f, r - r_f)^T \tag{67}$$

that is, the relative velocities including the antisymmetric terms from the gradient matrix $\boldsymbol{\Phi}$. The vector $\boldsymbol{F}_e$ represents any other external forces and moments that are applied to the vehicle such as gravity.

We will now consider small perturbations about a steady nonrotating flight condition. Let the vehicle state vector be given by

$$\boldsymbol{x} = \boldsymbol{x}_0 + \delta\boldsymbol{x} \tag{68}$$

and the fluid motion state by

$$\boldsymbol{x}_f = \boldsymbol{x}_{f0} + \delta\boldsymbol{x}_f \tag{69}$$

so that

$$\boldsymbol{x}_r = \boldsymbol{x}_{r0} + \delta\boldsymbol{x}_r \tag{70}$$

and

$$\delta\boldsymbol{x}_r = \delta\boldsymbol{x} + \delta\boldsymbol{x}_f \tag{71}$$

The initial relative velocities are

$$\boldsymbol{x}_{r0} = [U_{r0}, V_{r0}, W_{r0}, 0, 0, 0]^T \tag{72}$$

and combining the fluids bulk and circulating motions we write

$$\boldsymbol{x}_r = \boldsymbol{x} - \boldsymbol{x}_f - \boldsymbol{x}_c = \boldsymbol{x} - \boldsymbol{x}_g \tag{73}$$

so that

$$\begin{aligned}\boldsymbol{x}_0 &= [U_{r0} + U_{g0}, V_{r0} + V_{g0}, W_{r0} + W_{g0}, 0, 0, 0]^T \\ &= [U_0, V_0, W_0, 0, 0, 0]^T\end{aligned} \tag{74}$$

The perturbed equation (52) is

$$\begin{aligned}(\boldsymbol{M}_r + \boldsymbol{M}_i)\delta\dot{\boldsymbol{x}} &= (\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\delta\dot{\boldsymbol{x}}_f \times \boldsymbol{F}_{e0} + \delta\boldsymbol{F}_e + \boldsymbol{A}_v\left\{\boldsymbol{x}_{r0}^* + \delta\boldsymbol{x}_r^*\right\} \\ &\quad -(\delta\boldsymbol{P} + \boldsymbol{W}_0 + \delta\boldsymbol{W})(\boldsymbol{M}_i - \bar{\boldsymbol{M}}_i)(\boldsymbol{x}_0 + \delta\boldsymbol{x}) \\ &\quad - \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{B} & \boldsymbol{0}\end{bmatrix}\left(\delta\boldsymbol{P} + \boldsymbol{W}_{r0}^b + \delta\boldsymbol{W}_r^b + \begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\right) \\ &\quad \times\left(\boldsymbol{M}_r^b + \bar{m}\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\right)\left(\boldsymbol{x}_{r0}^b + \delta\boldsymbol{x}_r^b\right)\end{aligned} \tag{75}$$

where for small perturbations the nonperfect fluid forces and moments are given by

$$\boldsymbol{A}_v\left\{\boldsymbol{x}_{r0}^* + \delta\boldsymbol{x}_r^*\right\} = \boldsymbol{A}_v\left\{\boldsymbol{x}_{r0}^*\right\} + \boldsymbol{A}_e\delta\boldsymbol{x}_r^* \tag{76}$$

and $\boldsymbol{A}_e$ is the small perturbation aerodynamic derivative matrix for the steady-state flight condition. Remember that this only contains the nonperfect fluid terms at this stage.

Because we have perturbed about a steady state, we can write Eq. (75) as

$$\begin{aligned}(\boldsymbol{M}_r + \boldsymbol{M}_i)\delta\dot{\boldsymbol{x}} &= (\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\delta\dot{\boldsymbol{x}}_f \times \delta\boldsymbol{F}_e + \boldsymbol{A}_e\delta\boldsymbol{x}_r^* \\ &\quad -(\delta\boldsymbol{P} + \delta\boldsymbol{W})(\boldsymbol{M}_i - \bar{\boldsymbol{M}}_i)(\boldsymbol{x}_0) - \boldsymbol{W}_0(\boldsymbol{M}_i - \bar{\boldsymbol{M}}_i)\delta\boldsymbol{x} \\ &\quad - \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{B} & \boldsymbol{0}\end{bmatrix}\left\{\left(\delta\boldsymbol{P} + \delta\boldsymbol{W}_r^b\right)\left(\boldsymbol{M}_r^b + \bar{m}\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\right)\boldsymbol{x}_{r0}^b \right. \\ &\quad \left. + \left(\boldsymbol{W}_{r0}^b + \begin{bmatrix} \boldsymbol{\Phi} & 0 \\ 0 & 0\end{bmatrix}\right)\left(\boldsymbol{M}_r^b + \bar{m}\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\right)\delta\boldsymbol{x}_r^b\right\}\end{aligned} \tag{77}$$

The perfect fluid gradient effects can be seen to be due to the term

$$-\begin{bmatrix} \boldsymbol{\Phi} & 0 \\ 0 & 0\end{bmatrix}\left(\boldsymbol{M}_r^b + \bar{m}\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\right)\delta\boldsymbol{x}_r^b \tag{78}$$

that is, a set of forces only, acting through the center of buoyancy. Because these are just the perfect fluid effects, the symmetric part of the gradient matrix is used; the antisymmetric part and its effects are contained in $\boldsymbol{A}_e\delta\boldsymbol{x}_r^*$.

When we make the usual symmetry assumptions for airships and airplanes and substitute, the preceding expressions provide the small perturbation equations:

$$\begin{aligned}(\boldsymbol{M}_r + \boldsymbol{M}_i)\delta\dot{\boldsymbol{x}} &= (\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\delta\dot{\boldsymbol{x}}_g + \delta\boldsymbol{F}_e + \boldsymbol{A}_e\delta\boldsymbol{x}_g + \boldsymbol{A}_e\delta\boldsymbol{x} - \begin{bmatrix} \boldsymbol{\Phi} & 0 \\ 0 & 0\end{bmatrix}(\boldsymbol{M}_r + \bar{\boldsymbol{M}}_i)\delta\boldsymbol{x}_r \\ &+ \begin{bmatrix} 0 & 0 & 0 & 0 & -W_0(m-\bar{m}) & V_0(m-\bar{m}) \\ 0 & 0 & 0 & W_0(m-\bar{m}) & 0 & -U_0(m-\bar{m}) \\ 0 & 0 & 0 & -V_0(m-\bar{m}) & U_0(m-\bar{m}) & 0 \\ 0 & 0 & 0 & -W_0 a_z m & 0 & U_0 a_z m \\ 0 & 0 & 0 & -V_0 a_x m & -U_0 a_x m - W_0 a_z m & V_0 a_z m \\ 0 & 0 & 0 & -W_0 a_x m & 0 & -U_0 a_x m \end{bmatrix}\delta\boldsymbol{x} \\ &+ \begin{bmatrix} 0 & 0 & 0 & 0 & -W_{r0}(\bar{m} - Z_{\dot{w}}) & V_{r0}(\bar{m} - Y_{\dot{v}}) \\ 0 & 0 & 0 & W_{r0}(\bar{m} - Z_{\dot{w}}) & 0 & -U_{r0}(\bar{m} - X_{\dot{u}}) \\ 0 & 0 & 0 & -V_{r0}(\bar{m} - Y_{\dot{v}}) & U_{r0}(\bar{m} - X_{\dot{u}}) & 0 \\ 0 & -W_{r0}(Y_{\dot{v}} - Z_{\dot{w}}) & V_{r0}(Z_{\dot{w}} - Y_{\dot{v}}) & -W_{r0}Y_{\dot{p}} & V_{r0}(N_{\dot{v}} + Z_{\dot{q}}) & -U_{r0}M_{\dot{u}} - W_{r0}(M_{\dot{w}} + Y_{\dot{r}}) \\ W_{r0}(X_{\dot{u}} - Z_{\dot{w}}) & 0 & U_{r0}(X_{\dot{u}} - Z_{\dot{w}}) & -V_{r0}N_{\dot{v}} & -U_{r0}Z_{\dot{q}} + W_{r0}X_{\dot{q}} & V_{r0}L_{\dot{v}} \\ -V_{r0}(X_{\dot{u}} - Y_{\dot{v}}) & -U_{r0}(X_{\dot{u}} - Y_{\dot{v}}) & 0 & U_{r0}(M_{\dot{u}} + Y_{\dot{p}}) + W_{r0}M_{\dot{w}} & -V_{r0}(L_{\dot{v}} + X_{\dot{q}}) & U_{r0}Y_{\dot{r}} \end{bmatrix}\delta\mathbf{x}_r\end{aligned} \tag{79}$$

These are the standard small perturbation equations for airships,[26] but with unsteady fluid motion and velocity gradients included. Remember also that they include the perfect fluid effects mentioned in Sec. III.C.

If the displaced mass is put to zero and all of the added masses ignored, apart from $Z_{\dot{w}}$ and $M_{\dot{w}}$, then the standard small perturbation equations for the motion of airplanes in gusts are obtained (see Refs. 8 and 9), including the effects of fluid motion and gradients.

V. Conclusions

A new formulation of the equations of motion of a rigid vehicle in an unsteady heavy fluid has been derived that avoids the problems of earlier sets of equations used for submersibles, airships, and airplanes, particularly with regard to the moving fluid case. This is achieved by clearly separating out the inertias, the added masses, and the relative velocity effects and, by so doing, removing some of the problems encountered in the past. In addition, velocity gradients are taken into account in a way that links both irrotational and rotational effects.

Classic perfect fluid results are recovered and the source of error in an existing parafoil model is also revealed. The small perturbation equations revert to those that are normally used for both buoyant and lifting vehicles, but with the addition of fluid motion and gradient terms. As a result, the formulation provides a common set of equations of motion for describing the motion of underwater vehicles, airships, parafoils, and airplanes in a moving fluid.

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